OPTIMUM SHAPE FOR SINGLE EMISSION ELEMENTS

Yu. G. Zhulev

Inzhenerno-Fizicheskii Zhurnal, Vol. 13, No. 4, pp. 468–475, 1967

UDC 536.2.01:62-1

Optimum laws of variation in cross-sectional areas cooled by emission from solid and hollow radiation elements of various shapes are found.

1. Let us consider the problem of designing an optimum radiation-cooled heat-conduction pin of mini-

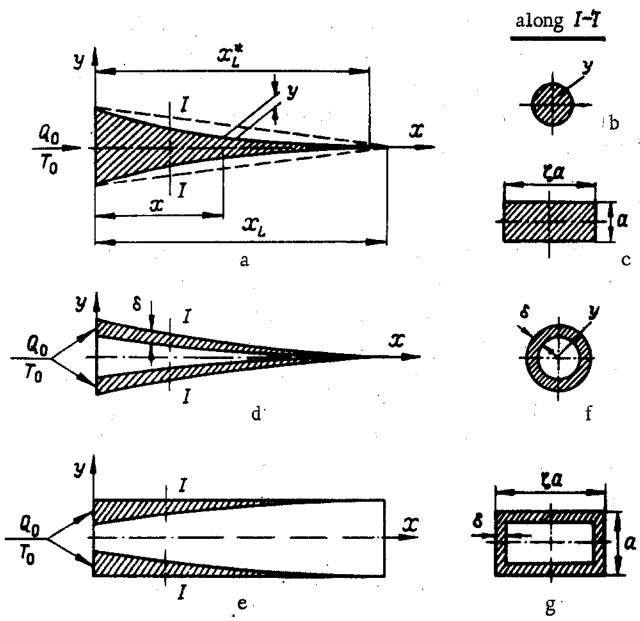

Fig. 1. Various shapes for single emitting elements: a) longitudinal cross section of heat-conducting pin; b and c) examples of its cross sections; d) longitudinal cross section of hollow pin; e) longitudinal cross section of heat-conducting element with constant perimeter; f and g) examples of lateral cross sections of a hollow pin and of a heat-conducting element with a constant perimeter.

mum weight whose area and cross-sectional perimeter would be defined by the equalities (Fig. 1a):

$$F = k_1 y^2; \quad \Pi = k_2 y. \tag{1}$$

We will consider rather long pins for which the equation of heat transfer along the pin and the law governing thermal radiation are valid in the following form:

$$k_1 y^2 \lambda \frac{dT}{dx} = -Q, \tag{2}$$

$$k_2 y \varepsilon\sigma T^4 dx = -dQ. \tag{3}$$

The optimum relationship y(x) must ensure minimum pin volume

$$V = \int_0^{x_L} k_1 y^2 dx \tag{4}$$

for given initial heat flux Q_0 and temperature T_0 (Fig. 1a).

We introduce the nondimensional variables

$$\bar{Q} = \frac{Q}{Q_0}; \quad \bar{T} = \frac{T}{T_0}; \quad \bar{x} = x \Big/ \left(\frac{Q_0 k_1 \lambda}{\varepsilon^2 k_2^2 \sigma^2 T_0^7}\right)^{1/3};$$

$$\bar{y} = y \Big/ \left(\frac{Q_0^2}{\varepsilon k_1 k_2 \lambda T_0^5 \sigma}\right)^{1/3}; \quad \bar{V} = V \Big/ \left(\frac{Q_0^5 k_1^2}{\lambda \sigma^4 \varepsilon^4 k_2^4 T_0^{17}}\right)^{1/3}. \tag{5}$$

In these variables Eqs. (2)–(4) are written in the form:

$$\bar{y}^2 \frac{d\bar{T}}{d\bar{x}} = -\bar{Q}; \tag{6}$$

$$\bar{y}\bar{T}^4 d\bar{x} = -d\bar{Q}; \tag{7}$$

$$\bar{V} = \int_0^{\bar{x}_L} \bar{y}^2 d\bar{x}. \tag{8}$$

At the end of the pin when $\bar{x} = \bar{x}_L$ we know only the value of the variable $\bar{Q} = 0$, and it is therefore advisable to express all of the variables as a function of $\bar{Q}$. Expression (8) is then rewritten as

$$\bar{V} = -\int_1^0 \left(\frac{\bar{Q}}{\bar{T}^{16}\frac{d\bar{T}}{d\bar{Q}}}\right)^{1/3} d\bar{Q} \tag{9}$$

and the formulated problem will correspond to the sought minimum of the functional (9) whose Euler equation will have the form

$$\frac{d^2\bar{T}}{d\bar{Q}^2} + \frac{16}{\bar{T}}\left(\frac{d\bar{T}}{d\bar{Q}}\right)^2 - \frac{1}{4\bar{Q}}\frac{d\bar{T}}{d\bar{Q}} = 0. \tag{10}$$

The general solution of this equation is written in the form

$$\bar{T} = C_1 (\bar{Q}^{\frac{5}{4}} + C_2)^{\frac{1}{17}}. \tag{11}$$

For the optimum pin contour it follows from the natural limit boundary condition at the right-hand end that $C_2 = 0$. At the base of the pin we must have $\bar{Q} = 1$ and $\bar{T} = 1$, and therefore $C_1 = 1$.

Thus considering (6)–(8), we have the following relationships characterizing the optimum pin:

$$\bar{T} = \bar{Q}^{\frac{5}{68}}; \quad \bar{x} = 2.58(1 - \bar{Q}^{\frac{33}{204}});$$

$$\bar{T} = \left(1 - \frac{\bar{x}}{2.58}\right)^{\frac{130}{561}}; \tag{12}$$

$$\bar{y} = 2.39\left(1 - \frac{\bar{x}}{2.58}\right)^{\frac{37}{11}}; \quad \bar{V}_{opt} = 1.91. \qquad (12)$$
(cont'd)

The various transverse cross-sectional shapes will be defined by the coefficients k_1 and k_2. For example, having substituted $k_1 = \pi$ and $k_2 = 2\pi$ into the above-cited expressions, we obtain the relationships for an optimum pin with a circular transverse cross section (Fig. 1b). Solution (12) coincides in this case with the result derived in [1] in the optimization of pins with circular transverse cross section and an exponential distribution of temperature along the plane. If $k_1 = \zeta$ and $k_2 = 2(1 + \zeta)$, we will have a pin with a rectangular cross section and a side ratio equal to ζ (Fig. 1c), etc.

We note that all of the results obtained above are also valid for hollow pins (Fig. 1d) in which there is no radiative heat exchange between the inside surfaces, and the wall thickness δ changes according to a definite law along the pin. For example, for a circular pin (Fig. 1f) the ratio $(y - \delta)/y$ must be constant. Here

$$k_1 = \pi\left[1 - \left(\frac{y-\delta}{y}\right)^2\right]; \quad k_2 = \pi.$$

For a pin with a rectangular profile whose sides are a and ζa long (Fig. 1g) it is necessary that

$$\left(1 - \frac{2\delta}{a}\right)\left(\zeta - \frac{2\delta}{a}\right) = \text{const.}$$

Here

$$k_1 = \left[\zeta - \left(1 - \frac{2\delta}{a}\right)\left(\zeta - \frac{2\delta}{a}\right)\right];$$

$$k_2 = 2(1 + \zeta).$$

We also note that the effectiveness of the optimum pin is independent of the shape of its lateral cross section

$$\Theta = Q_0 \Big/ \int_0^{x_{L\,opt}} k_2 y\, \varepsilon\sigma T_0^4\, dx = 0.706. \qquad (13)$$

However, the ratio of the removed flux to the weight of the pin is a strong function of the cross-sectional shape of the pin

$$\frac{Q_0}{G_p} = \frac{1}{1.91 \cdot \gamma}\left(\frac{\lambda\sigma^4\varepsilon^4 T_0^{17}}{Q_0^2}\right)^{\frac{1}{3}}\left(\frac{k_2^4}{k_1^2}\right)^{\frac{1}{3}}. \qquad (14)$$

Relationship (14) shows in particular that when the values of Q_0, λ, and T_0 are fixed, a solid pin with a lateral cross section in the form of a circle exhibits the smallest value for Q_0/G_p.

The results obtained above pertain to extremely pointed pins (when $x = x_{L\,opt}$, $y = 0$ and the first three derivatives of y with respect to x are equal to zero), as well as to the zero temperature T_L. Let us examine the class of optimum pins for $T_L \neq 0$. Here the function $\bar{T}(\bar{Q})$ is written as

$$\bar{T} = \left[(1 - \bar{T}_L^{17})\bar{Q}^{5/4} + \bar{T}_L^{17}\right]^{\frac{1}{17}}, \qquad (15)$$

while the contour shape is defined by the following system of equations:

$$\bar{y}^3 = \frac{(68/5)\bar{Q}^{3/4}\left[(1 - \bar{T}_L)\bar{Q}^{5/4} + \bar{T}_L^{17}\right]^{\frac{12}{17}}}{(1 - \bar{T}_L^{17})}; \qquad (16)$$

$$\bar{x} = -(68/5)^{-\frac{1}{3}} \times$$

$$\times \int_1^{\bar{Q}} \bar{Q}^{-\frac{1}{4}}\left[(1 - \bar{T}_L^{17})\bar{Q}^{5/4} + \bar{T}_L^{17}\right]^{-\frac{24}{51}}(1 - \bar{T}_L^{17})^{\frac{1}{3}}\, d\bar{Q}. \quad (17)$$

The effect of T_L on the volume of the optimum pin is

$$\bar{V} = \int_0^{\bar{x}_L} \bar{y}^2\, d\bar{x} = \frac{\bar{V}_{opt}}{(1 - \bar{T}_L^{17})}. \qquad (18)$$

The results obtained in calculating the optimum contours for pins with various values of $\bar{T}_L$ are shown in Fig. 2 (Sidorenko did the calculations). We note that when $\bar{T}_L \neq 0$ near the end of the pin, the derived solutions must naturally be refined, since the condition of a flattened contour is not satisfied there.

Let us compare the considered optimum pins with conical pins (the dashed lines in Fig. 1a). The area and perimeter of the lateral cross section in this case will be determined by Eqs. (1) in which y is defined in terms of x:

$$y = (L - x)\operatorname{tg}\frac{\alpha}{2}. \qquad (19)$$

With consideration of (19), Eqs. (2) and (3) lead to the differential equation

$$(\bar{L} - \bar{x}_c)\frac{d^2\bar{T}}{d\bar{x}_c^2} - 2\frac{d\bar{T}}{d\bar{x}_c} - N\bar{T}^4 = 0, \qquad (20)$$

where

$$\bar{x}_c = \frac{x}{x_L^*}; \quad \bar{L} = \frac{x_L}{x_L^*}; \quad N = \frac{\varepsilon\sigma T_0^3 x_L^* k_2}{\lambda k_1 \operatorname{tg}\frac{\alpha}{2}} \qquad (21)$$

(x_L^* is the length of the truncated cone, see Fig. 1a). The boundary conditions for Eq. (20) may be assumed to be

$$\bar{T} = 1 \text{ when } \bar{x}_c = 0; \quad \frac{d\bar{T}}{d\bar{x}_c} = 0 \quad \text{as} \quad \bar{L} \to 1. \quad (22)$$

A solution for (20) was derived numerically on a computer. The effectiveness of the conical pin as a function of N

$$\Theta_c = Q_0\left[\int_0^{x_L^*} k_2 (L - x)\operatorname{tg}\frac{\alpha}{2}\, \varepsilon\sigma T_0^4\, dx\right]^{-1} \qquad (23)$$

is shown in Fig. 3 (Potapov did the calculations). These relationships are valid for conical pins having any lateral cross-sectional shape (if there is no selfirradiation) and these relationships remain in force as well for hollow conical pins in which there is no

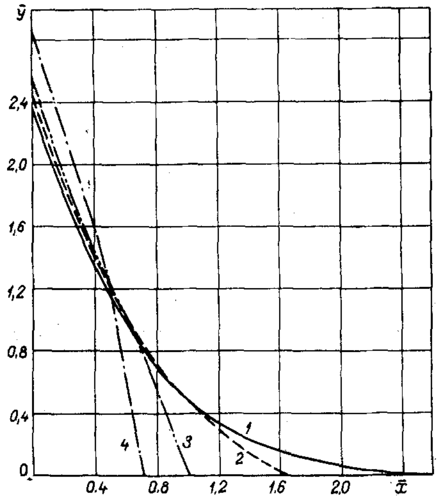

Fig. 2. Optimum pin contours for various values of $\overline{T}_L$: 1) $\overline{T}_L = 0$; 2) 0.8; 3) 0.9; 4) 0.95.

radiative heat exchange between the inside surfaces, and in which the wall thickness δ varies along the pin according to a specific law. For example, for a circular conical pin we must have

$$1-\frac{\delta}{(L-x)\operatorname{tg}\frac{\alpha}{2}}=\text{const}.$$

Here

$$k_1=\pi\left\{1-\left[1-\frac{\delta}{(L-x)\operatorname{tg}\frac{\alpha}{2}}\right]^2\right\};\quad k_2=2\pi.$$

Optimizing the dimensions of the conical pin, as was done by means of the function shown in Fig. 3, demonstrated that $N_{opt} = 0.6$ and that for identical values of Q_0, T_0, ε, and λ, as well as of k_1 and k_2, the optimum conical pin is heavier by only 1.5% than the optimum pin.

2. Let us consider the problem of designing the heat-conducting radiating element shown in Fig. 1e whose area and perimeter of lateral cross section would be defined by the equalities

$$F=k_3\psi;\quad \Pi=k_4, \tag{24}$$

where ψ is a function which depends on the law governing the change in wall thickness δ along the x-axis.

We will assume that there is no transfer of heat between the inside surfaces. The heat-transfer equation for the subject emitter with a constant outside lateral cross-sectional perimeter and the law of thermal emission are written in the form

$$k_3\psi\lambda\frac{dT}{dx}=-Q; \tag{25}$$

$$k_4\varepsilon\sigma T^4\,dx=-dQ. \tag{26}$$

The optimum function $\psi(x)$ must ensure a minimum volume for the subject emitter

$$V=\int_0^{x_L}k_3\psi\,dx \tag{27}$$

for given initial heat flux Q_0 and temperature T_0 (Fig. 1e).

We introduce the nondimensional variables

$$\tilde{x}=x\Big/\left(\frac{Q_0}{\varepsilon\sigma T_0^4k_4}\right),$$

$$\tilde{\psi}=\psi\Big/\left(\frac{Q_0^2}{\lambda k_3k_4\varepsilon\sigma T_0^5}\right);$$

$$\overline{V}=V\Big/\left(\frac{Q_0^3}{\lambda\varepsilon^2\sigma^2T_0^9k_4^2}\right). \tag{28}$$

In these variables (25)–(27) are written in the form

$$\tilde{\psi}\frac{d\overline{T}}{d\tilde{x}}=-\overline{Q}; \tag{29}$$

$$\overline{T}^4\,d\tilde{x}=-d\overline{Q}; \tag{30}$$

$$\tilde{V}=\int_0^{\tilde{x}_L}\tilde{\psi}\,d\tilde{x}. \tag{31}$$

At the end of the element (when $\tilde{x} = \tilde{x}_L$) we know the value only of the variable $\overline{Q} = 0$, and we will therefore express all of the variables as a function of $\overline{Q}$. After this expression (31) is rewritten as

$$\tilde{V}=-\int_1^0\frac{Qd\overline{Q}}{\frac{d\overline{T}}{d\overline{Q}}\overline{T}^8} \tag{32}$$

and the formulated problem will correspond to the seeking of a minimum for the function (32) whose Euler equation will have the form

$$2\overline{Q}\,\overline{T}\frac{d^2\overline{T}}{d\overline{Q}^2}+16Q\left(\frac{d\overline{T}}{d\overline{Q}}\right)^2-\overline{T}\frac{d\overline{T}}{d\overline{Q}}=0. \tag{33}$$

Equation (33) corresponds in terms of form and boundary conditions to the condition derived in [2] and which must be satisfied for a solitary infinite longitudinal fin of optimum shape. Using the results of [2], we will

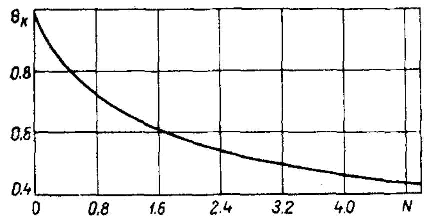

Fig. 3. Effectiveness of conical pin as a function of heat-conduction parameter N.

therefore derive the following relationships characterizing the optimum shape of the subject element:

$$\bar{T}=\bar{Q}^{1/6};\quad \bar{Q}=\left(1-\frac{\tilde{x}}{3}\right)^{3};\quad \bar{T}=\left(1-\frac{\tilde{x}}{3}\right)^{1/2};$$

$$\tilde{\psi}=6\left(1-\frac{\tilde{x}}{3}\right)^{3,5};\quad \tilde{V}_{\text{opt}}=4;\quad \tilde{x}_{L\,\text{opt}}=3. \qquad (34)$$

The various shapes for the lateral cross section of the subject element will be defined by the coefficients k_3 and k_4 and the form of the function $\psi(x)$. For example, if the subject emitting element is a tube having an outside radius y (Fig. 1f),

$$k_3=\pi y^2;\quad k_4=2\pi y;\quad \psi(x)=1-\left(\frac{y-\delta}{y}\right)^2.$$

It is not difficult to prove that the condition $\delta \le y$ in this case is expressed by the relationship

$$6Q_0^2 / 2\pi^2 y^3 \lambda\varepsilon\sigma T_0^5 \leqslant 1. \qquad (35)$$

However, if the subject element is rectangular in cross and if the relationship of its sides is expressed by ζ (Fig. 1g), we will have

$$k_3=a^2;\quad k_4=2a(1+\zeta);$$

$$\psi=\left[\zeta-\left(1-\frac{2\delta}{a}\right)\left(\zeta-\frac{2\delta}{a}\right)\right].$$

The effectiveness of the subject element is independent of the shape of the lateral cross section

$$\Theta_{\text{em}}=Q_0\Big/\int\limits_0^{x_{L\,\text{opt}}} k_4\varepsilon\sigma T_0^4\,dx=0.333. \qquad (36)$$

The derived results pertain to extremely pointed right-hand emitter edges (when $x = x_{L\,\text{opt}}$, $\psi(x)$ and the first three derivatives of ψ with respect to x vanish), as well as to the zero temperature $\bar{T}_L$. However, if in Fig. 2 of [2] we take $\tilde{\psi}$ instead of $\bar{y}$ and assume that $\tilde{x} = -\bar{x}$, we will obtain the functions for the construction of the subject emitters when $\bar{T}_L \neq 0$. In particular, the effect of $\bar{T}_L$ on the volume of an emitter with an optimum law $\delta(x)$ will be expressed by the relationship

$$\tilde{V}_{\text{opt}\,(\bar{T}_{L\neq 0})}=\frac{\tilde{V}_{\text{opt}\,(\bar{T}_{L=0})}}{(1-\bar{T}_L)^{9}}. \qquad (37)$$

The weight ratio for the optimum emitters considered in sections 1 and 2 for identical values of Q_0, T_0, λ, ε and identical shapes of lateral cross sections and dimensions with x = 0 will be defined as

$$\frac{G_{\text{em}}}{G_{\text{p}}}=\frac{2.09}{k_4^2}\left(\frac{Q_0^4}{\lambda^2\sigma^2\varepsilon^2 T_0^{10}}\right)^{1/3}\left(\frac{k_2^4}{k_1^2}\right)^{1/3}=0.37. \qquad (38)$$

Thus an emitter with a constant outside lateral cross-sectional perimeter is considerably lighter than the optimum pin.

NOTATION

Here F and Π are the area and external cross-section circumference of the radiating element at a distance x from its base; y is the quantity depending on x and having the dimension of length; ψ is a dimensionless value; k_1 and k_2 are dimensionless quantities independent of x; k_3 and k_4 are dimensional quantities independent of x (k_3 has the dimension of area, k_4 of length); Q_0 is the heat flux from the radiating element; Q is the heat flux along the element at a distance x from its base; λ is the thermal conductivity; T_0, T, and T_L are the temperatures at the element base, at a distance x from the base and at the right-hand end of the element; ε is the emissivity of the surface; σ is the Stefan—Boltzmann constant; V is the radiator volume (V_{opt} is the volume of the optimum radiatior); δ is the wall thickness; x_L is the radiator length ($x_{L\,\text{opt}}$ is the length of the optimum radiator); γ is the specific weight; Θ, Θ_c, and Θ_{ra} are the effectiveness of optimum pin, conical pin, and radiator with constant external circumference; G_p and G_{em} are the weight of the pin and the emitter with constant external circumference; α is the angle between the generatrices of the conic surface in the longitudinal section of the pin by a plane; $\bar{T}$ and $\bar{Q}$ are the dimensionless temperature and flow rate, respectively; $\bar{x}$, $\bar{y}$, and $\bar{V}$ are dimensionless variables in the study of optimum pins of minimum weight; x_c and N are the dimensionless variable and parameter of thermal conductivity with conical pins; $\tilde{x}$, $\tilde{\psi}$, and $\tilde{V}$ are dimensionless variables for the emitter with constant external circumference; a and ζa are the dimensions of the sides of the radiator cross-section.

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